2.4 Circular Waveguide

BYU ECEn-462

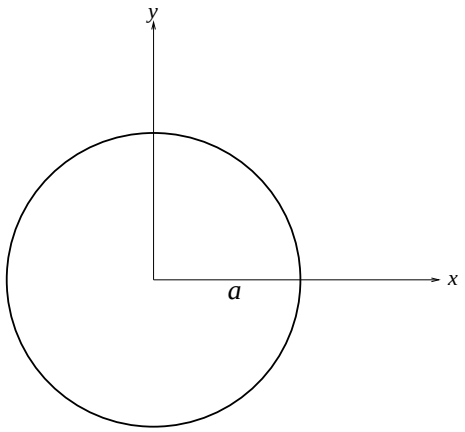

Figure 2.5: A circular waveguide of radius a.

For a circular waveguide of radius a (Fig. 2.5), we can perform the same sequence of steps in cylindrical coordinates as we did in rectangular coordinates to find the transverse field components in terms of the longitudinal (i.e. E_z, H_z) components. In cylindrical coordinates, the transverse field is

$$\overline{E}_T = \hat{\rho}E_\rho + \hat{\phi}E_\phi \qquad \overline{H}_T = \hat{\rho}H_\rho + \hat{\phi}H_\phi \tag{2.66}$$

Using this in Maxwell's equations (where the curl is applied in cylindrical coordinates) leads to

$$H_\rho = \frac{j}{k_c^2}\left(\frac{\omega\epsilon}{\rho}\frac{\partial E_z}{\partial \phi} - \beta\frac{\partial H_z}{\partial \rho}\right) \tag{2.67}$$

$$H_\phi = \frac{-j}{k_c^2}\left(\omega\epsilon\frac{\partial E_z}{\partial \rho} - \frac{\beta}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{2.68}$$

$$E_\rho = \frac{-j}{k_c^2}\left(\beta\frac{\partial E_z}{\partial \rho} - \frac{\omega\mu}{\rho}\frac{\partial H_z}{\partial \phi}\right) \tag{2.69}$$

$$E_\phi = \frac{-j}{k_c^2}\left(\frac{\beta}{\rho}\frac{\partial E_z}{\partial \phi} - \omega\mu\frac{\partial H_z}{\partial \rho}\right) \tag{2.70}$$

where $k_c^2 = k^2 - \beta^2$ as before. Please note that here (as well as in rectangular waveguide derivation), we have assumed $e^{-j\beta z}$ propagation. For $e^{+j\beta z}$ propagation, we replace β with $-\beta$.

2.4.1 TE Modes

We don't need to prove that the wave travels as $e^{\pm j\beta z}$ again since the differentiation in z for the Laplacian is the same in cylindrical coordinates as it is in rectangular coordinates ($\partial^2/\partial z^2$). However, the ρ and ϕ derivatives of the Laplacian are different than the x and y derivatives. The wave equation for H_z is

$$(\nabla^2 + k^2)H_z = 0 \tag{2.71}$$

$$\left(\frac{\partial^2}{\partial \rho^2} + \frac{1}{\rho}\frac{\partial}{\partial \rho} + \frac{1}{\rho^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2} + k^2\right)H_z(\rho,\phi,z) = 0 \tag{2.72}$$

Using the separation of variables approach, we let $H_z(\rho,\phi,z) = R(\rho)P(\phi)e^{-j\beta z}$, and obtain

$$\left[R''P + \frac{1}{\rho}R'P + \frac{1}{\rho^2}RP'' + \underbrace{(k^2-\beta^2)}_{k_c^2}RP\right]e^{-j\beta z} = 0 \tag{2.73}$$

Multiplying by a common factor leads to

$$\underbrace{\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + \rho^2k_c^2}_{\text{function of }\rho} + \underbrace{\frac{P''}{P}}_{\text{function of }\phi} = 0 \tag{2.74}$$

Because the terms in this equation sum to a constant, yet each depends only on a single coordinate, each term must be constant:

$$\frac{P''}{P} = -k_\phi^2 \qquad \rightarrow \qquad P'' + k_\phi^2 P = 0 \tag{2.75}$$

so that

$$P(\phi) = A_0\sin(k_\phi\phi) + B_0\cos(k_\phi\phi) \tag{2.76}$$

Using this result in (2.74) leads to

$$\rho^2\frac{R''}{R} + \rho\frac{R'}{R} + (\rho^2k_c^2 - k_\phi^2) = 0 \tag{2.77}$$

or

$$\boxed{\rho^2R'' + \rho R' + (\rho^2k_c^2 - k_\phi^2)R = 0} \tag{2.78}$$

This is known as *Bessel's Differential Equation*.

Now, we could use the Method of Frobenius to solve this equation, but we would just be repeating a well-known solution. The series you obtain from such a solution has very special properties (a lot like sine and cosine: you may recall that $\sin(x)$ and $\cos(x)$ are really just shorthand for power series that have special properties).

The solution is

$$R(\rho) = C_0J_{k_\phi}(k_c\rho) + D_0N_{k_\phi}(k_c\rho) \tag{2.79}$$

where $J_\nu(x)$ is the Bessel function of the first kind of order ν and $N_\nu(x)$ is the Bessel function of the second kind of order ν.

1. First, let's examine k_ϕ.

$$H_z(\rho,\phi,z) = \left[C_0J_{k_\phi}(k_c\rho) + D_0N_{k_\phi}(k_c\rho)\right]\left[A_0\sin(k_\phi\phi) + B_0\cos(k_\phi\phi)\right]e^{-j\beta z} \tag{2.80}$$

Clearly, $H_z(\rho,\phi,z) = H_z(\rho,\phi+2\pi\ell,z)$ where ℓ is an integer. This can only be true if $\boxed{k_\phi = \nu}$, where $\nu =$ integer.

$$H_z(\rho,\phi,z) = \left[C_0J_\nu(k_c\rho) + D_0N_\nu(k_c\rho)\right]\left[A_0\sin(\nu\phi) + B_0\cos(\nu\phi)\right]e^{-j\beta z} \tag{2.81}$$

2. It turns out that $N_\nu(k_c\rho) \to -\infty$ as $\rho \to 0$. Clearly, $\rho = 0$ is in the domain of the waveguide. Physically, however, we can't have infinite field intensity at this point. This leads us to conclude that $\boxed{D_0 = 0}$. We now have

$$H_z(\rho, \phi, z) = [A\sin(\nu\phi) + B\cos(\nu\phi)]\, J_\nu(k_c\rho)e^{-j\beta z} \tag{2.82}$$

3. The relative values of A and B have to do with the absolute coordinate frame we use to define the waveguide. For example, let $A = F\cos(\nu\phi_0)$ and $B = -F\sin(\nu\phi_0)$ (you can find a value of F and ϕ_0 to make this work). Then

$$A\sin(\nu\phi) + B\cos(\nu\phi) = F\sin[\nu(\phi - \phi_0)] \tag{2.83}$$

The value of ϕ_0 that makes this work can be thought of as the *coordinate reference* for measuring ϕ. So, we really are left with finding F, which is simply the mode amplitude and is therefore determined by the excitation.

4. We still need to determine k_c. The boundary condition that we can apply is $E_\phi(a, \phi, z) = 0$, where $\rho = a$ represents the waveguide boundary. Since

$$E_\phi(\rho, \phi, z) = \frac{j\omega\mu}{k_c^2}\frac{\partial H_z}{\partial\rho} \tag{2.84}$$

$$= \frac{j\omega\mu}{k_c^2}[A\sin(\nu\phi) + B\cos(\nu\phi)]\, k_c J'_\nu(k_c\rho)e^{-j\beta z} \tag{2.85}$$

where

$$J'_\nu(x) = \frac{d}{dx}J_\nu(x), \tag{2.86}$$

our boundary condition indicates that $J'_\nu(k_c a) = 0$. So

$$k_c a = p'_{\nu n} \qquad \to \qquad k_c = \frac{p'_{\nu n}}{a} \tag{2.87}$$

where $p'_{\nu n}$ is the nth zero of $J'_\nu(x)$. Below is a table of a few of the zeros of $J'_\nu(x)$:

$J'_\nu(k_c a) = 0$	$n = 1$	$n = 2$	$n = 3$
$\nu = 0$	0.0000	3.8317	7.0156
$\nu = 1$	1.8412	5.3314	8.5363
$\nu = 2$	3.0542	6.7061	9.9695

5. We have already defined $k_c^2 = k^2 - \beta^2$, so

$$\beta^2 = k^2 - \left(\frac{p'_{\nu n}}{a}\right)^2 \tag{2.88}$$

Note that there is no "ϕ" term here. However, the ϕ variation of the fields in the waveguide does influence β. (How?)

6. Cutoff frequency ($\beta = 0$): Since $k = k_c = 2\pi f_{c,\nu n}/c$ at the mode cutoff frequency,

$$f_{c,\nu n} = \frac{c}{2\pi}\frac{p'_{\nu n}}{a} \tag{2.89}$$

7. Dominant Mode: We don't count the $\nu = 0$, $n = 1$ mode (TE_{01}) since $p'_{01} = 0$ resulting in zero fields. The dominant TE mode is therefore the mode with the smallest non-zero value of $p'_{\nu n}$, which is the TE_{11} mode.

8. The expressions for wavelength and phase velocity derived for the rectangular waveguide apply here as well. However, you must use the proper value for the cutoff frequency in these expressions.

2.4.2 TM Modes

The derivation is the same except that we are solving for E_z. We can therefore write

$$E_z(\rho, \phi, z) = [A \sin(\nu\phi) + B \cos(\nu\phi)]\, J_\nu(k_c\rho) e^{-j\beta z} \tag{2.90}$$

Our boundary condition in this case is $E_z(a, \phi, z) = 0$ or $J_\nu(k_c a) = 0$. This leads to

$$k_c a = p_{\nu n} \qquad \rightarrow \qquad k_c = \frac{p_{\nu n}}{a} \tag{2.91}$$

where $p_{\nu n}$ is the nth zero of $J_\nu(x)$.

$J_\nu(k_c a) = 0$	$n = 1$	$n = 2$	$n = 3$
$\nu = 0$	2.4048	5.5201	8.6537
$\nu = 1$	3.8317	7.0156	10.1735
$\nu = 2$	5.1356	8.4172	11.6198

In this case, we have

$$\beta^2 = k^2 - \left(\frac{p_{\nu n}}{a}\right)^2 \tag{2.92}$$

$$f_{c,\nu n} = \frac{c}{2\pi}\frac{p_{\nu n}}{a} \tag{2.93}$$

It becomes clear the the TE_{11} mode is the dominant overall mode of the waveguide.

2.4.3 Bessel Functions

Here are some of the basic properties of Bessel functions:

$$J_\nu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n (x/2)^{2n+\nu}}{n!(n+\nu)!} \tag{2.94}$$

$$N_\nu(x) = \lim_{p\to\nu} \frac{J_p(x)\cos(p\pi) - J_{-p}(x)}{\sin(p\pi)} \tag{2.95}$$

$$J_\nu(-x) = (-1)^\nu J_\nu(x), \quad \nu = \text{integer} \tag{2.96}$$

$$J_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \cos(x - \pi/4 - \nu\pi/2), \quad x \to \infty \tag{2.97}$$

$$N_\nu(x) \simeq \sqrt{\frac{2}{\pi x}} \sin(x - \pi/4 - \nu\pi/2), \quad x \to \infty \tag{2.98}$$

$$\frac{d}{dx} Z_\nu(x) = Z_{\nu-1}(x) - \nu Z_\nu(x)/x \tag{2.99}$$

where Z is any Bessel function. Figures 2.6 and 2.7 show Bessel functions of the first and second kinds of orders 0, 1, 2, 3.

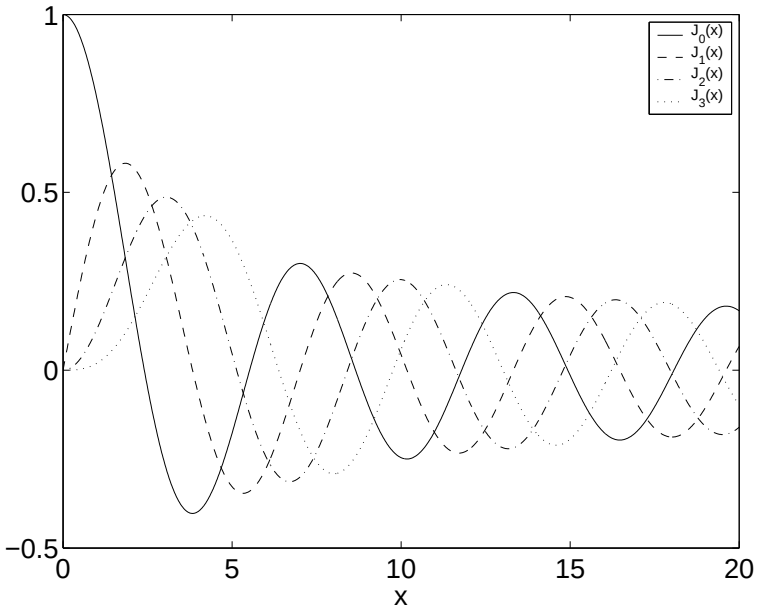

Figure 2.6: Bessel functions of the first kind.

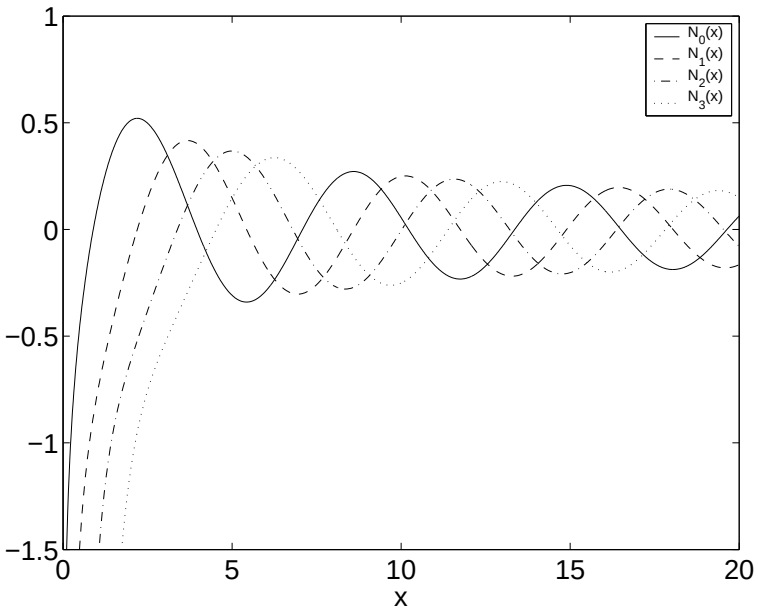

Figure 2.7: Bessel functions of the second kind.